PMM U.S.S.R.,Vol.50,No.1,pp.119-122,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

THE USE OF QUADRATIC FORMS TO STUDY THE STABILITY WITH RESPECT TO PART OF THE VARIABLES*

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The conditions for the existence of the Lyapunov function in the form of a quadratic form with constant coefficients, positive definite with respect to a part of the variables, necessary for the non-linear systems, and necessary and sufficient for linear stationary systems, are obtained. Examples are considered.

1. Let us consider a system of differential equations of the perturbed motion /1/

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}) \quad (\mathbf{X}(t, 0) \equiv 0) \tag{1.1}$$
$$\mathbf{x} = (y_1, \ldots, y_m, z_1, \ldots, z_p)^T, \ m > 0, \ p \geqslant 0, \ n = m + p$$

We assume that: a) the right-hand sides of system (1.1) are continuous in the region

$$t \geqslant 0, \ \|\mathbf{y}\| \leqslant H > 0, \ \|\mathbf{z}\| < +\infty \tag{1.2}$$

and satisfy the conditions of uniqueness of the solution $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$, determined by the initial conditions $\mathbf{x}(t_0; t_0, \mathbf{x}_0) = \mathbf{x}_0$, b) the solutions of system (1.1) are z-continuable.

Let us assume that the k $(1 \leqslant k \leqslant n)$ first integrals are known for system (1.1)

$$V_i(t, \mathbf{x}) = \mathrm{const} \ (V_i(t, 0) \equiv 0), \ i = 1, \ldots, k \tag{1.3}$$

Pozharitskii /2/ obtained the necessary and sufficient conditions for the existence of the positive definite function

$$V(t, \mathbf{x}) = F(V_1(t, \mathbf{x}), \ldots, V_k(t, \mathbf{x})) \tag{1.4}$$

formed from the integrals (1.3). The results can be extended to embrace the problem of stability with respect to part of the variables, namely:

1°. The necessary and sufficient condition for any y-positive definite function (1.4) to exist is, that the function

$$V_0(t, \mathbf{x}) \equiv F_0(V_1(t, \mathbf{x}), \ldots, V_k(t, \mathbf{x})) = V_1^2(t, \mathbf{x}) + \ldots + V_k^2(t, \mathbf{x}) \tag{1.5}$$

be y-positive definite.

2°. The function (1.5) is y-positve definite if and only if for at least one of the integrals, say $V_i(t, \mathbf{x})$, a pair of functions $\mu_i(r) \in K$ and $\nu_i(r) \in K$ exists such that $V_i^2(t, \mathbf{x}) \geqslant \mu_i(\|\mathbf{y}\|)$ as long as $V_1^2 + \ldots + V_{i-1}^2 + V_{i+1}^2 + \ldots + V_k^2 \leqslant \nu_i(\|\mathbf{y}\|)$ (the function $a(r) \in K$ by definition /3/, provided that $a(r)$ is continuous and increases monotonically, and $a(0) = 0$).

We note /2/ that if such a pair of functions can be found for one integral, then it can be found for any other integral.

In /4/, for the case of analytic, time-independent integrals (1.3), the conditions enabling these integrals to be used to construct a function whose expansion in a Maclaurin' series begins with a positive-definite quadratic form were obtained. The results obtained in /4/ cannot be extended to the problem of stability with respect to a part of the variables, since we cannot, generally speaking, use the y-positive definiteness of the quadratic part of the Lyapunov function to draw any sort of conclusions concerning the sign definiteness (with respect to all or some of the variables) of the function itself.

In the connection it is useful to establish the conditions (both necessary and sufficient) for the existence, in system (1.1), of the Lyapunov function as a quadratic form with constant coefficients.

Here and henceforth the function $V(t, \mathbf{x})$ will be called the Lyapunov function if /1, 5/ $V(t, \mathbf{x}) \geqslant a(\|\mathbf{y}\|)$ and $V^{\cdot}(t, \mathbf{x}) \leqslant 0$ by virtue of system (1.1).

2. *Theorem 1.* The quadratic form $v(\mathbf{x})$ with constant coefficients will be y-positive definite if and only if a number $\lambda > 0$ exists for which $v(\mathbf{x})$ can be written in the form

$$v(\mathbf{x}) = \lambda(y_1^2 + \ldots + y_m^2) + \xi_1^2 + \ldots + \xi_k^2, \ 1 \leqslant k \leqslant n \tag{2.1}$$

$$\xi_i = \alpha_{i1}y_1 + \ldots + \alpha_{im}y_m + \beta_{i1}z_1 + \ldots + \beta_{ip}z_p \ (i = 1, \ldots, n; \alpha_{ij}, \beta_{ij} - \mathrm{const}) \tag{2.2}$$

Proof. The sufficiency of the condition is obvious. We shall show the necessity. Let $v(\mathbf{x}) \geqslant a(\|\mathbf{y}\|)$, $a(r) \in K$. We write $\lambda = a(1) > 0$, then $v(\mathbf{x}) \geqslant \lambda$ on the surface $\|\mathbf{y}\| = 1$. We put every

*Prikl.Matem.Mekhan.,50,1,163-167,1986

point $\mathbf{x} = (\mathbf{y}, \mathbf{z})^T$ with $\mathbf{y} \neq 0$ in 1:1 correspondence with the points $\mathbf{x}^* = (\mathbf{y}^*, \mathbf{z}^*)^T$ according to the rule $\mathbf{x} = \mathbf{x}^* \| \mathbf{y} \|$. Here clearly $\| \mathbf{y}^* \| = 1$, therefore $v(\mathbf{x}^*) \geqslant \lambda$. Since for the quadratic form we have $v(c\mathbf{x}) = c^2 v(\mathbf{x})$ $(c = \text{const})$, we obtain $v(\mathbf{x}) = v(\mathbf{x}^* \| \mathbf{y} \|) = \| \mathbf{y} \|^2 v(\mathbf{x}^*) \geqslant \lambda \| \mathbf{y} \|^2$ from which it follows that $v(\mathbf{x}) - \lambda (y_1^2 + \ldots + y_m^2) \geqslant 0$. A non-negative quadratic form appears on the left-hand side of this inequality. As we know /6/, it can be reduced to the form (2.1) by means of the linear non-degenerate transformation (2.2).

The proof implies that if the function $v(\mathbf{x})$ can be written in the form (2.1), (2.2) for some $\lambda = \lambda_0 > 0$, then it can be written in the same form for any $\lambda \in (0, \lambda_0)$ and the constants α_{ij}, β_{ij} and k are dependent on λ.

Theorem 2. If a quadratic form $v(\mathbf{x})$ with constant coefficients exists, representing a Lyapunov function for the system (1.1), then the system has a q-dimensional $(0 \leqslant q \leqslant p)$, positively invariant, uniformly stable subspace situated in $\mathbf{R}_z^p = \{\mathbf{x}: \mathbf{y} = 0\}$.

The subspace will be given in explicit form below.

Proof. According to Theorem 1 the function $v(\mathbf{x})$ can be written in the form (2.1), (2.2). Let us consider the set $M = \{\mathbf{x}: v(\mathbf{x}) = 0\}$. According to (2.1), (2.2) we have

$$M = \{\mathbf{x}: \mathbf{y} = 0,\ \beta_{i1} z_1 + \ldots + \beta_{ip} z_p = 0\ (i = 1, \ldots, k)\} \tag{2.3}$$

We shall show that M satisfies the required conditions.

Clearly, M is a subspace and its dimension is $q = p - \operatorname{rank} \| \beta_{ij} \| (i = 1, \ldots, k;\ j = 1, \ldots, p)$, $M \subset \mathbf{R}_z^p$, $0 \leqslant q \leqslant p$. If $\mathbf{x}_0 \in M$, then $v(\mathbf{x}_0) = 0$; since $v \geqslant 0$, we have $v(\mathbf{x}(t; t_0, \mathbf{x}_0)) \equiv 0$ for all $t \geqslant t_0$ and hence /7/, $\mathbf{x}(t; t_0, \mathbf{x}_0) \in M$ for any $t \geqslant t_0$, i.e. M is positively invariant. It can be shown that the form (2.1), (2.2) is positive definite with respect to the distance $\rho(\mathbf{x}, M)$ from the set M. Since v is independent of t and $v^\cdot \leqslant 0$ by virtue of the system (1.1), it follows /8/ that the invariant set (2.3) is uniformly stable. The theorem is proved.

Clearly, the uniform stability of the set (2.3) implies, at any q, $0 \leqslant q \leqslant p$, the uniform y-stability of the motion $\mathbf{x} = 0$. If $\operatorname{rank} \| \beta_{ij} \| = p$, then $q = 0$, the set M contracts to the point $\mathbf{x} = 0$ and $v(\mathbf{x})$ is positive definite. In this case the motion $\mathbf{x} = 0$ is uniformly positive in all variables (in particular in $\mathbf{y}$). If on the other hand $\beta_{ij} = 0$ $(\operatorname{rank} \| \beta_{ij} \| = 0)$, then $q = p$ and the set M becomes the subspace $\mathbf{R}_z^p$, in which case the quadratic form (2.1), (2.2) will depend only on $\mathbf{y}$ and $\mathbf{R}_z^p$ will be uniformly stable /1, 9/.

Note. The function

$$v(\mathbf{x}) = w(\mathbf{y}) + \xi_1^2 + \ldots + \xi_k^2 \tag{2.4}$$

where $w(\mathbf{y})$, is a y-positive definite quadratic form and the functions ξ_i are defined in (2.2), can be reduced to a form analogous to (2.1). Indeed, the function (2.4) differs from (2.1) by the following additional term on the right-hand side:

$$w(\mathbf{y}) - \lambda (y_1^2 + \ldots + y_m^2) \tag{2.5}$$

For sufficiently small $\lambda > 0$ the quadratic form (2.5) is y-positive definite, and can therefore be reduced to the sum of squares of forms linear in y_i. It is clear that the sets (2.3) identical for the functions (2.1) and (2.4).

Theorem 2 yields the necessary conditions for the existence of the Lyapunov function $v(\mathbf{x})$. We find that for the linear system

$$\begin{aligned} y_i^\cdot &= \sum_{j=1}^{m} a_{ij} y_j + \sum_{l=1}^{p} b_{il} z_l \quad (i = 1, \ldots, m) \\ z_s^\cdot &= \sum_{j=1}^{m} c_{sj} y_j + \sum_{l=1}^{p} d_{sl} z_l \quad (s = 1, \ldots, p) \end{aligned} \tag{2.6}$$

with constant coefficients these conditions are also sufficient (the stability of autonomous systems is always uniform; therefore in what follows we shall omit, for brevity, any mention of uniformity).

Theorem 3. If the zero solution of system (2.6) is y-stable, a quadratic form exists with constant coefficients which is the Lyapunov function for this system.

Proof. If all $b_{il} = 0$, then the theorem is obvious. Let us assume that a $b_{il} \neq 0$ exists. Then, as was shown in /10/, we can use new variables

$$\mu_i = \sum_{l=1}^{p} \beta_{il} z_l \quad (i = 1, \ldots, r;\ r \leqslant p) \tag{2.7}$$

where β_{il} is expressed in terms of the coefficients of system (2.6), and transform to the μ-form

$$\begin{aligned} y_i^\cdot &= \sum_{j=1}^{m} a_{ij} y_j + \sum_{l=1}^{r} e_{il} \mu_l \quad (i = 1, \ldots, m) \\ \mu_s^\cdot &= \sum_{j=1}^{m} a_{sj}^* y_j + \sum_{l=1}^{r} e_{sl}^* \mu_l \quad (s = 1, \ldots, r) \end{aligned} \tag{2.8}$$

and the zero solution of system (2.6) will be y-stable if and only if the zero solution of system (2.8) is stable in all variables y, μ. The stability of the zero solution of system (2.8) implies [11-13] the existence of a positive-definite quadratic form $w(y, \mu)$ such, that $w^{\cdot}_{(2.8)} \leqslant 0$. By virtue of the properties of the function w, there exists $\lambda > 0$ such that $w(y, \mu) \geqslant \lambda (\| y \|^2 + \| \mu \|^2) \geqslant \lambda \| y \|^2$. Therefore, replacing the arguments μ_i in the function $w(y, \mu)$ according to (2.7), we obtain y-positive definite quadratic form $v(y, z)$, and $v^{\cdot}_{(2.6)} \leqslant 0$. The theorem is proved.

Corollary 1. If the zero solution of the system (2.6) is y-stable, then the system has a q-dimensional $(0 \leqslant q \leqslant p)$ positively invariant stable subspace $M \subset \mathbf{R}_z^p$.

The corollary follows from Theorem 2 and 3. We note that $M = \{(y, \mu): w(y, \mu) = 0\}$, therefore $M = \{x: y = 0, \beta_{i1}z_1 + \ldots + \beta_{ip}z_p = 0 \ (i = 1, \ldots, r)\}$, where β_{il} is the coefficient given by (2.7) and $q = p - r$.

Corollary 2. The following assertions are equivalent for system (2.6):

1°. The zero solution is y-stable.

2°. Every solution is y-bounded.

3°. The zero solution of the system in μ-form corresponding to the system (2.6) is stable.

4°. There exists a q-dimensional $(0 \leqslant q \leqslant p)$ positively invariant stable subspace $M \subset \mathbf{R}_z^p$.

5°. There exists a Lyapunov function in the form of a quadratic form with constant coefficients.

The equivalence $1° \Leftrightarrow 2°$ was proved in /14/ and $1° \Leftrightarrow 3°$ in /10/. The relations $1° \Leftrightarrow 4°$ and $1° \Leftrightarrow 5°$ are obtained from Theorems 2 and 3 and Corollary 1.

If the conditions of Theorem 2 hold, then according to /15/ the set (2.3), positively invariant by virtue of system (1.1), is uniformly stable under constant perturbations integrally small in the neighbourhood of this set. If in addition $v^{\cdot}_{(1.1)}$ is negative definite in $\rho(x, M)$, then the set (2.3) is uniformly stable under the constantly acting perturbations small in the mean or small at every instant of time near this set. We also note that if a y-positive quadratic form $v(x)$ exists such that $v^{\cdot}_{(2.6)}$ is negative definite in $\rho^2(x, M)$, then /12, 16/ the set (2.3) is positively invariant and exponentially asymptotically stable in the linear approximation for any system

$$y_i^{\cdot} = \sum_{j=1}^{m} a_{ij}y_j + \sum_{l=1}^{p} b_{il}z_l + Y_i(t, y, z) \quad (i = 1, \ldots, m)$$

$$z_s^{\cdot} = \sum_{j=1}^{m} c_{ij}y_j + \sum_{l=1}^{p} d_{sl}z_l + Z_s(t, y, z) \quad (s = 1, \ldots, p)$$

in which $\| Y(t, y, z) \| + \| Z(t, y, z) \| \leqslant A\rho(x, M)$ with sufficiently small constant $A > 0$.

3. *Example* 1. /5, 17/. A heavy rigid body with one fixed point, in the Lagrange's case, admits of an infinite set of permanent rotations. The equations of perturbed motion have the following first integrals (in the notation of /5/):

$$V_1 = A(\xi_1^2 + \xi_2^2 + 2p_0\xi_1 + 2q_0\xi_2) + C(\xi_3^2 + 2r_0\xi_3) + 2Pz_0\eta_3$$
$$V_2 = A(p_0\eta_1 + \alpha\xi_1 + q_0\eta_2 + \beta\xi_2 + \xi_1\eta_1 + \xi_2\eta_2) + C(r_0\eta_3 + \gamma\xi_3 + \xi_3\eta_3)$$
$$V_3 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2(\alpha\eta_1 + \beta\eta_2 + \gamma\eta_3), \quad V_4 = \xi_3$$

The function /5/ $V = V_1 - 2\omega V_2 + A\omega^2 V_3 + C\mu V_4^2$ is reduced to the form

$$V = C(1 + \mu)\xi_3^2 - 2C\omega\xi_3\eta_3 + A\omega^2\eta_3^2 + A(\xi_1 - \omega\eta_1)^2 + A(\xi_2 - \omega\eta_2)^2 \tag{3.1}$$

There exists $\mu > 0$ such that the quadratic form $C(1 + \mu)\xi_3^2 - 2C\omega\xi_3\eta_3 + A\omega^2\eta_3^2$ is positive definite. Therefore the function (3.1) is positive definite in ξ_3, η_3 and given in a form analogous to that of (2.4). Using Theorem 2 and the note made in Sect.2, we conclude that the equations of perturbed motion have an invariant stable set

$$\xi_3 = \eta_3 = 0, \ \xi_1 - \omega\eta_1 = 0, \ \xi_2 - \omega\eta_2 = 0 \tag{3.2}$$

lying in the subspace $\xi_3 = \eta_3 = 0$. From this follows, in particular, the stability of the unperturbed motion with respect to ξ_3, η_3 /5/. We note that the invariant set (3.2) corresponds to the subset of the set of permanent rotations.

Example 2 [18]. The motion of mechanical holonomic system with normal coordinates $x_1, \ldots, x_n$, acted upon by dissipative gyroscopic forces and radial correction forces is described, without the non-linear terms, by the system of equations

$$x_i^{\cdot\cdot} = -b_i x_i + \sum_{j=1}^{n} g_{ij}x_j^{\cdot} + \sum_{j=1}^{n} e_{ij}x_j \quad (i = 1, \ldots, n; g_{ij} = -g_{ji}; e_{ij} = -e_{ji}) \tag{3.3}$$

Let us write $e_{ij} = \alpha g_{ij} \ (\alpha > 0)$ and consider the function /18/

$$V = \frac{1}{2}\sum_{i=1}^{n} x_i^{\cdot 2} + \frac{1}{2}\alpha\sum_{i=1}^{n} b_i x_i^2 + \alpha\sum_{i=1}^{n} x_i x_i^{\cdot} \tag{3.4}$$

Its time derivative is, by virtue of system (3.3)

$$V^{\cdot} = -\sum_{i=1}^{n}(b_i - \alpha)\, x_i^{\cdot 2}.$$

Let /18/ $b_1 = \ldots = b_m = \alpha,\ b_{m+1} > \alpha, \ldots, b_n > \alpha.$ In this case $V^{\cdot} \leqslant 0$ and the function (3.4) will become

$$V = \left\{\frac{1}{2}\sum_{i=m+1}^{n} x_i^{\cdot 2} + \frac{1}{2}\alpha\sum_{i=m+1}^{n} b_i x_i^2 + \alpha\sum_{i=m+1}^{n} x_i x_i^{\cdot}\right\} + \frac{1}{2}\sum_{i=1}^{n}(x_i^{\cdot} + \alpha x_i)^2$$

The quadratic form appearing within the braces is positive definite in $x_{m+1}, \ldots, x_n, x_{m+1}^{\cdot}, \ldots, x_n^{\cdot}$. Taking into account the note from Sect.2 and using Theorem 2, we conclude that system (3.3) has a positively invariant stable set

$$x_i = x_i^{\cdot} = 0,\ x_j^{\cdot} + \alpha x_j = 0\ (i = m+1, \ldots, n;\ j = 1, \ldots, m)$$

(this remains valid in the case when $g_{ij} = g_{ij}(\mathbf{x}),\ e_{ij} = \alpha g_{ij}$).

This implies, in particular, the stability of the unperturbed motion in $x_{m+1}, \ldots, x_n, x_{m+1}^{\cdot}, \ldots, x_n^{\cdot}$ /18/. It can be shown that the stability with respect to $x_{m+1}^{\cdot}, \ldots, x_n^{\cdot}$ is in this case asymptotic.

Example 3 /10/. Let us consider a linear, stationary, third-order system

$$y^{\cdot} = -y + z_1 - 2z_2,\quad z_1^{\cdot} = 4y + z_1,\quad z_2^{\cdot} = 2y + z_1 - z_2 \tag{3.5}$$

and y-positive definite quadratic form $v = y^2 + (z_1 - 2z_2)^2$. Its derivative will be, by virtue of system (3.5) $v^{\cdot} = -y^2 - (y - z_1 + 2z_2)^2 - (z_1 - 2z_2)^2$.

Using Theorem 2, we conclude that system (3.5) has a positively invariant stable set

$$\{(y, z_1, z_2):\ y = 0,\ z_1 - 2z_2 = 0\} \tag{3.6}$$

Remembering that $v^{\cdot} \leqslant -v$, we can show that the set (3.6) is exponentially asymptotically stable. This in particular implies the exponentially asymptotic y-stability of the zero solution of system (3.5) /10/.

The author thanks V.V. Rumyantsev for his interest.

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Translated by L.K.